Computers and Mathematics with Applications 62 (2011) 1215–1227

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Existence of solutions for a class of nonlinear Volterra singular integral equations

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ARTICLE INFO

Keywords:
Volterra singular integral equation
Measure of noncompactness
Fixed point theorem
Function of bounded variation
Integral equation of Volterra–Stieltjes type

ABSTRACT

In this paper we prove some results concerning the existence of solutions for a large class of nonlinear Volterra singular integral equations in the space $C[0, 1]$ consisting of real functions defined and continuous on the interval $[0, 1]$. The main tool used in the proof is the concept of a measure of noncompactness. We also present some examples of nonlinear singular integral equations of Volterra type to show the efficiency of our results. Moreover, we compare our theory with the approach depending on the use of the theory of Volterra–Stieltjes integral equations. We also show that the results of the paper are applicable in the study of the so-called fractional integral equations which are recently intensively investigated and find numerous applications in describing some real world problems.

1. Introduction

Singular integral equations play important role in applied problems. Numerous problems in mathematical physics, mechanics and engineering are linked with the theory of singular integral equations [1–3] and several research papers studied those integral equations [4–15].

In this paper we will consider nonlinear Volterra singular integral equation of the form

$$x(t) = f_1(t, x(t), x(a(t))) + (Gx)(t) \int_0^t f_2(t, s)(Qx)(s)ds, \tag{1.1}$$

where $t \in I = [0, 1]$, $a : I \to I$ is a continuous function and G, Q are some operators acting continuously from the space $C(I)$ into itself. The symbol $C(I)$ stands for the space consisting of all real functions defined and continuous on the interval I. Moreover, we assume that the function f_2 has the form

$$f_2(t, s) = k(t, s)g(t, s), \tag{1.2}$$

where $k : \Delta \to \mathbb{R}$ is continuous and g is monotonic with respect to the first variable and may be discontinuous on the triangle $\Delta = \{(t, s) : 0 \le s \le t \le 1\}$.

Let us mention that for $f_1(t, x, y) = b(t), f_2(t, s) = k(t, s)/(t - s)^{1-\alpha} (0 < \alpha < 1)$ and $(Gx)(t) = f(t, x)$, Eq. (1.1) was studied in [16] and for $f_1(t, x, y) = f(t, x), f_2(t, s) = k(t, s)/(t - s)^{1-\alpha} (0 < \alpha < 1)$ was considered in [7].

Recently a lot of authors used the concept of a measure of noncompactness in order to prove the existence of solutions for a wide variety of functional integral equations (cf. [4,6–8,11,16,17]). In this paper we also use that concept and a fixed point theorem of Darbo type to prove results on the existence of solutions of Eq. (1.1) belonging to the space $C(I)$.

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doi:10.1016/j.camwa.2011.03.049

In the final section of this paper we show that the results obtained here are parallel with those from the paper [6], where we considered a particular case of Eq. (1.1) with help of an approach associated with the theory of nonlinear integral equation of Volterra–Stieltjes type.

It is worthwhile mentioning that the results obtained in the paper can be applied to a wide class of the so-called fractional integral equations. Indeed, we show that some types of fractional integral equations are a special case of nonlinear singular integral equations investigated in the paper. Let us notice that fractional integral equations are recently intensively studied and they find a lot of interesting and nontrivial applications (cf. [4,6,12,18–22]).

Finally, let us remark that the results obtained in this paper generalize those from the papers [1,4,7–9,13,16], for example.

2. Preliminaries

In this section we provide some notation, definitions and auxiliary facts which will be needed further on.

Denote by $\mathbb{R}$ the set of real numbers and put $\mathbb{R}_+ = [0, \infty)$. Let $(E, \|\cdot\|)$ be a Banach space with zero element θ. The symbol $\overline{X}$, $\operatorname{Conv}X$ will denote the closure and closed convex hull of a subset X of E, respectively. Moreover, let $\mathfrak{M}_E$ indicate the family of all nonempty and bounded subsets of E and $\mathfrak{N}_E$ indicate the family of all nonempty and relatively compact sets.

We use the following definition of the measure of noncompactness given in [23].

Definition 2.1. A mapping $\mu : \mathfrak{M}_E \to \mathbb{R}_+$ is said to be a measure of noncompactness in E if it satisfies the following conditions:

1° The family $\ker\mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $\ker\mu \subset \mathfrak{N}_E$.
2° $X \subseteq Y \Rightarrow \mu(X) \le \mu(Y)$.
3° $\mu(\overline{X}) = \mu(X)$.
4° $\mu(\operatorname{Conv}X) = \mu(X)$.
5° $\mu(\lambda X + (1-\lambda)Y) \le \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0, 1]$.
6° If (X_n) is a sequence of closed sets from $\mathfrak{M}_E$ such that $X_{n+1} \subset X_n$ for $n = 1, 2, \ldots$ and if $\lim_{n\to\infty}\mu(X_n) = 0$ then the set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

In what follows we recall a fixed point theorem of Darbo type [23].

Theorem 2.2. *Let Ω be a nonempty, bounded, closed and convex subset of a space E and let $F : \Omega \to \Omega$ be a continuous mapping such that there exists a constant $k \in [0, 1)$ with the property $\mu(FX) \le k\mu(X)$ for any nonempty subset X of Ω. Then F has a fixed point in the set Ω.*

In the sequel we will work in the Banach space $C[a, b]$ consisting of real functions defined and continuous on the interval $[a, b]$ and endowed with the standard maximum norm. For simplicity we will assume that $[a, b] = I = [0, 1]$. Then the space in question is denoted by $C(I)$.

One of the most important and convenient measure of noncompactness in the space $C(I)$ can be defined in the following way presented below (cf. [23]).

Namely, take an arbitrary set $X \in \mathfrak{M}_{C(I)}$. For $x \in X$ and $\varepsilon > 0$ let us put

$$\omega(x, \varepsilon) = \sup\{|x(t) - x(s)| : t, s \in I, |t - s| \le \varepsilon\},$$
$$\omega(X, \varepsilon) = \sup\{\omega(x, \varepsilon) : x \in X\},$$
$$\omega_0(X) = \lim_{\varepsilon\to 0}\omega(X, \varepsilon). \tag{2.1}$$

It may be shown [23] that the mapping $\omega_0 = \omega_0(X)$ is the measure of noncompactness in the space $C(I)$ having some additional properties. For example, $\omega_0(\lambda X) = |\lambda|\omega_0(X)$ and $\omega_0(X+Y) \le \omega_0(X)+\omega_0(Y)$ for $X, Y \in \mathfrak{M}_{C(I)}$ and for $\lambda \in \mathbb{R}$ [23].

3. Main results

In this section we will investigate the solvability of the functional integral Eq. (1.1) in the space $C(I)$. We will assume that the following conditions are satisfied:

(i) $a : I \to I$ is a continuous function.
(ii) The function $f_1 : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a nonegative constant p such that
$$|f_1(t, x_1, y_1) - f_1(t, x_2, y_2)| \le p\max\{|x_1 - x_2|, |y_1 - y_2|\},$$
for any $t \in I$ and for all $x_1, x_2, y_1, y_2 \in \mathbb{R}$.
(iii) The operator G transforms continuously the space $C(I)$ into itself and there exists a nonnegative constant q such that for any set $X \in \mathfrak{M}_{C(I)}$ the following inequality holds
$$\omega_0(GX) \le q\omega_0(X).$$
(iv) There exists a nondecreasing function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that
$$\|Gx\| \le \varphi(\|x\|),$$
for any $x \in C(I)$.

(v) The operator Q acts continuously from the space $C(I)$ into itself and there exists a nondecreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|Qx\| \leq \psi(\|x\|),$$

for any $x \in C(I)$.

(vi) $f_2 : \Delta \to \mathbb{R}$ has the form (1.2), where the function $k : \Delta \to \mathbb{R}$ is continuous.

(vii) The function $g(t, s) = g : \Delta \to \mathbb{R}_+$ occurring in decomposition (1.2) is monotonic with respect to t (on the interval $[s, 1]$) and for any fixed $t \in I$ the function $s \to g(t, s)$ is Lebesgue integrable over the interval $[0, t]$. Moreover, for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $t_1, t_2, \in I$ with $t_1 < t_2$ and $t_2 - t_1 \leq \delta$ the following inequalities are satisfied:

$$\left| \int_0^{t_1} [g(t_2, s) - g(t_1, s)]\mathrm{d}s \right| \leq \varepsilon, \tag{3.1}$$

$$\int_{t_1}^{t_2} g(t_2, s)\mathrm{d}s \leq \varepsilon. \tag{3.2}$$

In what follows we discuss a few facts connected with assumption (vii) which plays the basic role in our considerations. First, let us consider the function $h : I \to \mathbb{R}_+$ defined in the following way

$$h(t) = \int_0^t g(t, s)\mathrm{d}s. \tag{3.3}$$

Obviously h is well-defined in view of assumption (vii).

Then we have the following lemma.

Lemma 3.1. *Under assumption* (vii) *the function h is continuous on the interval I.*

Proof. Fix $\varepsilon > 0$ arbitrarily and choose $\delta > 0$ to the number $\varepsilon/2$ according to assumption (vii). Next, take arbitrary numbers $t_1, t_2 \in I$ such that $|t_2 - t_1| \leq \delta$. Without loss of generality we may assume that $t_1 < t_2$. Then we get:

$$\begin{aligned} |h(t_2) - h(t_1)| &= \left| \int_0^{t_2} g(t_2, s)\mathrm{d}s - \int_0^{t_1} g(t_1, s)\mathrm{d}s \right| \\ &= \left| \int_0^{t_1} g(t_2, s)\mathrm{d}s - \int_0^{t_1} g(t_1, s)\mathrm{d}s + \int_{t_1}^{t_2} g(t_2, s)\mathrm{d}s \right| \\ &\leq \left| \int_0^{t_1} [g(t_2, s) - g(t_1, s)\mathrm{d}s]\mathrm{d}s \right| + \int_{t_1}^{t_2} g(t_2, s)\mathrm{d}s \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

and the proof is complete. $\square$

Remark 3.2. Observe that assumption (vii) can be formulated in other ways. Namely, assume that there is satisfied assumption (vii), where instead of (3.1) we require that the function h be continuous on I. Then the function $g(t, s)$ satisfies condition (3.1).

Indeed, take $\varepsilon > 0$ and choose a number $\delta_1 > 0$ such that for all $t_1, t_2 \in I$ with $t_1 < t_2$ and $t_2 - t_1 \leq \delta_1$ we have

$$\int_{t_1}^{t_2} g(t_2, s)\mathrm{d}s \leq \frac{\varepsilon}{2}.$$

Next, choose a number $\delta_2 > 0$ to the number $\varepsilon/2$ according to the continuity of the function h on the interval I. Put $\delta = \min\{\delta_1, \delta_2\}$. Then, for arbitrary $t_1, t_2 \in I$ such that $t_1 < t_2$ and $t_2 - t_1 \leq \delta$ we obtain

$$\begin{aligned} \left| \int_0^{t_1} [g(t_2, s) - g(t_1, s)]\mathrm{d}s \right| &= \left| \int_0^{t_2} g(t_2, s)\mathrm{d}s - \int_{t_1}^{t_2} g(t_2, s)\mathrm{d}s - \int_0^{t_1} g(t_1, s)\mathrm{d}s \right| \\ &\leq \left| \int_0^{t_2} g(t_2, s)\mathrm{d}s - \int_0^{t_1} g(t_1, s)\mathrm{d}s \right| + \left| \int_{t_1}^{t_2} g(t_2, s)\mathrm{d}s \right| \\ &= |h(t_2) - h(t_1)| + \int_{t_1}^{t_2} g(t_2, s)\mathrm{d}s \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

This completes the proof.

Finally we show that in the case when for any fixed $s \in I$ the function $t \to g(t, s)$ is nondecreasing on the interval $[s, 1]$, the assumption (vii) can be formulated equivalently in a more convenient form.

Lemma 3.3. *Assume that the function $g(t,s) = g : \Delta \to \mathbb{R}_+$ is nondecreasing with respect to t and for any fixed $t \in I$ the function $s \to g(t,s)$ is Lebesgue integrable over the interval $[0,t]$. Moreover, assume that the function $h = h(t)$ defined by formula (3.3) is continuous on I. Then the function g satisfies conditions (3.1) and (3.2).*

Proof. Fix $\varepsilon > 0$ arbitrarily and choose $\delta > 0$ according to the continuity of the function h, i.e. for arbitrary $t_1, t_2 \in I$, $t_1 < t_2$ and such that $t_2 - t_1 \le \delta$ we have

$$|h(t_2) - h(t_1)| \le \varepsilon. \tag{3.4}$$

Then we get

$$\begin{aligned} |h(t_2) - h(t_1)| &= \left| \int_0^{t_2} g(t_2,s)\mathrm{d}s - \int_0^{t_1} g(t_1,s)\mathrm{d}s \right| \\ &= \left| \int_{t_1}^{t_2} g(t_2,s)\mathrm{d}s + \int_0^{t_1} [g(t_2,s) - g(t_1,s)]\mathrm{d}s \right|. \end{aligned} \tag{3.5}$$

Keeping in mind the fact that the function $t \to g(t,s)$ is nondecreasing on I, we get

$$\int_0^{t_1} [g(t_2,s) - g(t_1,s)]\mathrm{d}s \ge 0. \tag{3.6}$$

Since the function g is nonnegative on the triangle Δ, we have

$$\int_{t_1}^{t_2} g(t_2,s)\mathrm{d}s \ge 0. \tag{3.7}$$

Now, linking (3.4)–(3.7), we obtain

$$|h(t_2) - h(t_1)| = \int_{t_1}^{t_2} g(t_2,s)\mathrm{d}s + \int_0^{t_2} [g(t_2,s) - g(t_1,s)]\mathrm{d}s \le \varepsilon.$$

Hence, in view of (3.6) and (3.7) we conclude that the function $g(t,s)$ satisfies both conditions (3.1) and (3.2).
The proof is complete. □

Observe that the converse assertion to that from the above lemma is contained in Lemma 3.1.

In order to formulate the last assumption needed in our considerations concerning Eq. (1.1), let us define the constants $\overline{k}, \overline{f_1}, \overline{h}$ by putting

$$\overline{k} = \sup\{|k(t,s)| : (t,s) \in \Delta\},$$
$$\overline{f_1} = \sup\{|f_1(t,0,0)| : t \in I\},$$
$$\overline{h} = \sup\{h(t) : t \in I\}.$$

Notice that the constants $\overline{k}$ and $\overline{f_1}$ are finite in view of assumptions (vi) and (ii), respectively, while the fact that $\overline{h} < \infty$ is a consequence of assumption (vii) and Lemma 3.1.

Now, we are in a position to formulate the announced assumption:

(viii) There exists a positive solution r_0 of the inequality

$$pr + \overline{f_1} + \overline{kh}\varphi(r)\psi(r) \le r,$$

such that $p + \overline{khq}\psi(r_0) < 1$.

In what follows we will consider the operators associated with Eq. (1.1) and defined on the space $C(I)$ in the following way:

$$(F_1x)(t) = f_1(t, x(t), x(a(t))),$$
$$(F_2x)(t) = \int_0^t f_2(t,s)(Qx)(s)\mathrm{d}s,$$
$$(Fx)(t) = (F_1x)(t) + (Gx)(t)(F_2x)(t),$$

for $t \in I$.

Moreover, for further purposes we introduce two functions M and N defined on $\mathbb{R}_+$ in the following way:

$$M(\varepsilon) = \sup\left\{ \left| \int_0^{t_1} [g(t_2,s) - g(t_1,s)]\mathrm{d}s \right| : t_1, t_2 \in I,\ t_1 < t_2,\ t_2 - t_1 \le \varepsilon \right\},$$
$$N(\varepsilon) = \sup\left\{ \int_{t_1}^{t_2} g(t_2,s)\mathrm{d}s : t_1, t_2 \in I,\ t_1 < t_2,\ t_2 - t_1 \le \varepsilon \right\}.$$

In view of assumption (vii) we have that $M(\varepsilon) \to 0$ and $N(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Then, we can state the following lemma.

Lemma 3.4. *Under assumptions* (i)–(vii) *the operator F transforms continuously the space $C(I)$ into itself.*

Proof. Fix arbitrarily a function $x \in C(I)$. Then, in view of the properties of the so-called superposition operator [24] we deduce that $F_1x \in C(I)$. On the other hand, taking arbitrary functions $x, y \in C(I)$ and employing assumption (ii), for a fixed $t \in I$ we get

$$\begin{aligned}|(F_1x)(t) - (F_1y)(t)| &= |f_1(t, x(t), x(a(t))) - f_1(t, y(t), y(a(t)))| \\ &\le p \max\{|x(t) - y(t)|, |x(a(t)) - y(a(t))|\}.\end{aligned}$$

By assumption (i) the above inequality yields

$$\|F_1x - F_1y\| \le p\|x - y\|. \tag{3.8}$$

Hence we conclude that F_1 acts continuously from the space $C(I)$ into itself.

Further, fix $x \in C(I)$ and $\varepsilon > 0$. Take $t_1, t_2 \in I$ such that $|t_2 - t_1| \le \varepsilon$. Without loss of generality we may assume that $t_1 < t_2$. Then, keeping in mind our assumptions, we derive the following estimates:

$$\begin{aligned}
&|(F_2x)(t_2) - (F_2x)(t_1)| \\
&\quad \le \left|\int_0^{t_2} k(t_2, s)g(t_2, s)(Qx)(s)\mathrm{d}s - \int_0^{t_1} k(t_2, s)g(t_2, s)(Qx)(s)\mathrm{d}s\right| \\
&\qquad + \left|\int_0^{t_1} k(t_2, s)g(t_2, s)(Qx)(s)\mathrm{d}s - \int_0^{t_1} k(t_1, s)g(t_2, s)(Qx)(s)\mathrm{d}s\right| \\
&\qquad + \left|\int_0^{t_1} k(t_1, s)g(t_2, s)(Qx)(s)\mathrm{d}s - \int_0^{t_1} k(t_1, s)g(t_1, s)(Qx)(s)\mathrm{d}s\right| \\
&\quad \le \int_{t_1}^{t_2} |k(t_2, s)|g(t_2, s)|(Qx)(s)|\mathrm{d}s + \int_0^{t_1} |k(t_2, s) - k(t_1, s)|g(t_2, s)|(Qx)(s)|\mathrm{d}s \\
&\qquad + \int_0^{t_1} |k(t_1, s)\|g(t_2, s) - g(t_1, s)\|(Qx)(s)|\mathrm{d}s \\
&\quad \le \bar{k}\psi(\|x\|)\int_{t_1}^{t_2} g(t_2, s)\mathrm{d}s + \omega_1(k, \varepsilon)\psi(\|x\|)\int_0^{t_1} g(t_2, s)\mathrm{d}s + \bar{k}\psi(\|x\|)\int_0^{t_1} |g(t_2, s) - g(t_1, s)|\mathrm{d}s \\
&\quad \le \bar{k}\psi(\|x\|)N(\varepsilon) + \omega_1(k, \varepsilon)\psi(\|x\|)\int_0^{t_2} g(t_2, s)\mathrm{d}s + \bar{k}\psi(\|x\|)\int_0^{t_1} |g(t_2, s) - g(t_1, s)|\mathrm{d}s,
\end{aligned}$$

where we denoted

$$\omega_1(k, \varepsilon) = \sup\{|k(t_2, s) - k(t_1, s)| : (t_1, s), (t_2, s) \in \Delta, |t_2 - t_1| \le \varepsilon\}.$$

Since the function $t \to g(t, s)$ is monotonic by assumption (vii), it is easy to check that

$$\int_0^{t_1} |g(t_2, s) - g(t_1, s)|\mathrm{d}s = \left|\int_0^{t_1} [g(t_2, s) - g(t_1, s)]\mathrm{d}s\right|.$$

Taking into account this fact and the estimate obtained above, we get

$$|(F_2x)(t_2) - (F_2x)(t_1)| \le \bar{k}\psi(\|x\|)N(\varepsilon) + \omega_1(k, \varepsilon)\psi(\|x\|)h(t_2) + \bar{k}\psi(\|x\|)\left|\int_0^{t_1} [g(t_2, s) - g(t_1, s)]\mathrm{d}s\right|.$$

This yields

$$\omega(F_2x, \varepsilon) \le \bar{k}\psi(\|x\|)N(\varepsilon) + \bar{h}\psi(\|x\|)\omega_1(k, \varepsilon) + \bar{k}\psi(\|x\|)M(\varepsilon), \tag{3.9}$$

where the functions $M(\varepsilon)$ and $N(\varepsilon)$ were defined earlier.

Observe that in view of the uniform continuity of the function k on the triangle Δ we have that $\omega_1(k, \varepsilon) \to 0$ as $\varepsilon \to 0$.

Combining this fact with the properties of the functions $M(\varepsilon)$ and $N(\varepsilon)$ established above and taking into account estimate (3.9) we infer that the function F_2x is continuous on I. This means that F_2 maps the space $C(I)$ into itself.

Consequently, keeping in mind that $F_1 : C(I) \to C(I)$ and assumption (iii) we conclude that the operator F transforms the space $C(I)$ into itself.

Now, we show that the operator F is continuous on the space $C(I)$. To do this fix arbitrarily $x_0 \in C(I)$ and $\varepsilon > 0$. Next, take $x \in C(I)$ such that $\|x - x_0\| \leq \varepsilon$. Then, for arbitrary $t \in I$ we obtain:

$$\begin{aligned} |(Fx)(t) - (Fx_0)(t)| &\leq |(F_1x)(t) - (F_1x_0)(t)| + |(Gx)(t)(F_2x)(t) - (Gx_0)(t)(F_2x_0)(x)| \\ &\leq \|F_1x - F_1x_0\| + |(Gx)(t)\|(F_2x)(t) - (F_2x_0)(t)| + |(F_2x_0)(t)\|(Gx)(t) - (Gx_0)(x)| \\ &\leq \|F_1x - F_1x_0\| + \|Gx\||(F_2x)(t) - (F_2x_0)(t)| + \|F_2x_0\|\|Gx - Gx_0\|. \end{aligned} \tag{3.10}$$

On the other hand we derive the following estimate:

$$\begin{aligned} |(F_2x)(t) - (F_2x_0)(t)| &= \left|\int_0^t k(t,s)g(t,s)(Qx)(s)\mathrm{d}s - \int_0^t k(t,s)g(t,s)(Qx_0)(s)\mathrm{d}s\right| \\ &\leq \int_0^t |k(t,s)|g(t,s)|(Qx)(s) - (Qx_0)(s)|\mathrm{d}s \\ &\leq \bar{k}\left(\int_0^t g(t,s)\mathrm{d}s\right)\|Qx - Qx_0\| \leq \overline{kh}\|Qx - Qx_0\|. \end{aligned} \tag{3.11}$$

Similarly, for arbitrarily fixed $t \in I$, we have

$$\begin{aligned} |(F_2x_0)(t)| &= \left|\int_0^t k(t,s)g(t,s)(Qx_0)(s)\mathrm{d}s\right| \\ &\leq \bar{k}\left(\int_0^t g(t,s)\mathrm{d}s\right)\|Qx_0\| \leq \overline{kh}\|Qx_0\|. \end{aligned}$$

Hence, we get

$$\|F_2x_0\| \leq \overline{kh}\|Qx_0\|. \tag{3.12}$$

Next, combining (3.10)–(3.12) and (3.8) we obtain the following estimate

$$\|Fx - Fx_0\| \leq p\|x - x_0\| + \|Gx\|\overline{kh}\|Qx - Qx_0\| + \overline{kh}\|Qx_0\|\|Gx - Gx_0\|.$$

Taking into account estimates from assumptions (iv) and (v), in view of the above inequality we derive the following estimate

$$\|Fx - Fx_0\| \leq p\varepsilon + \varphi(\|x_0\| + \varepsilon)\overline{kh}\|Qx - Qx_0\| + \overline{kh}\psi(\|x_0\|)\|Gx - Gx_0\|.$$

From the above estimate and the continuity of the operators G and Q required in assumptions (iii) and (v), we deduce that the operator F is continuous in the space $C(I)$.

The proof is complete. □

Now we present our main result.

Theorem 3.5. *Under assumptions* (i)–(viii) *Eq.* (1.1) *has at least one solution in the space* $C(I)$.

Proof. Fix $x \in C(I)$ arbitrarily and $t \in I$. Then, employing our assumptions and evaluating similarly as in the proof of Lemma 3.4, we get

$$\begin{aligned} |(Fx)(t)| &\leq |f_1(t,x(t),x(a(t))) - f_1(t,0,0)| + |f_1(t,0,0)| + |(Gx)(t)|\left|\int_0^t k(t,s)g(t,s)(Qx)(s)\mathrm{d}s\right| \\ &\leq p\|x\| + \overline{f_1} + \|Gx\|\overline{kh}\|Qx\| \\ &\leq \overline{f_1} + p\|x\| + \overline{kh}\varphi(\|x\|)\psi(\|x\|). \end{aligned}$$

This yields the following estimate

$$\|Fx\| \leq \overline{f_1} + p\|x\| + \overline{kh}\varphi(\|x\|)\psi(\|x\|).$$

From the above estimate and assumption (viii) we infer that there exists $r_0 > 0$ such that the operator F maps the ball B_{r_0} into itself and $p + \overline{kh}q\psi(r_0) < 1$. Moreover, in view of Lemma 3.4 the operator F is continuous on the ball B_{r_0}.

Now, let us fix a nonempty subset X of the ball B_{r_0} and a number $\varepsilon > 0$. Then, for an arbitrary $x \in X$ and for $t_1, t_2 \in I$ such that $|t_2 - t_1| \leq \varepsilon$, in virtue of (3.12) and the imposed assumptions, we obtain

$$\begin{aligned} |(Fx)(t_2) - (Fx)(t_1)| \leq\ & |f_1(t_2,x(t_2),x(a(t_2))) - f_1(t_1,x(t_2),x(a(t_2)))| \\ & + |f_1(t_1,x(t_2),x(a(t_2))) - f_1(t_1,x(t_1),x(a(t_1)))| \\ & + |(F_2x)(t_2)\|(Gx)(t_2) - (Gx)(t_1)| + |(Gx)(t_1)\|(F_2x)(t_2) - (F_2x)(t_1)| \\ \leq\ & \omega_{r_0}(f_1,\varepsilon) + p\max\{|x(t_2) - x(t_1)|, |x(a(t_2)) - x(a(t_1))|\} \\ & + \overline{kh}\psi(r_0)\omega(Gx,\varepsilon) + \varphi(r_0)\omega(F_2x,\varepsilon), \end{aligned}$$

where we denoted

$$\omega_{r_0}(f_1,\varepsilon)=\sup\{|f_1(t_2,x,y)-f_1(t_1,x,y)|:t_1,t_2\in I,|t_2-t_1|\le\varepsilon,\quad x,y\in[-r_0,r_0]\}.$$

Hence, in view of (3.9) we deduce the following estimate

$$\begin{aligned}\omega(Fx,\varepsilon)\le{}&\omega_{r_0}(f_1,\varepsilon)+p\max\{\omega(x,\varepsilon),\omega(x,\omega(a,\varepsilon))\}\\&+\overline{kh}\psi(r_0)\omega(Gx,\varepsilon)+\varphi(r_0)\psi(r_0)[\overline{k}N(\varepsilon)+\overline{h}\omega_1(k,\varepsilon)+\overline{k}M(\varepsilon)],\end{aligned}$$

where the quantity $\omega_1(k,\varepsilon)$ was introduced previously.

Now, taking into account the uniform continuity of the function f_1 on the set $I\times[-r_0,r_0]^2$ and the properties of the functions $k(t,s)$, $a(t)$, $M(\varepsilon)$ and $N(\varepsilon)$ established above or imposed in assumptions (i) and (vi), we obtain

$$\lim_{\varepsilon\to0}\omega_1(k,\varepsilon)=\lim_{\varepsilon\to0}\omega(a,\varepsilon)=\lim_{\varepsilon\to0}\omega_{r_0}(f_1,\varepsilon)=\lim_{\varepsilon\to0}M(\varepsilon)=\lim_{\varepsilon\to0}N(\varepsilon)=0.$$

Linking these statements with the above estimate and formula (2.1), we deduce the following inequality

$$\omega_0(FX)\le p\omega_0(X)+\overline{kh}\psi(r_0)\omega_0(GX).$$

Hence, in view of assumption (iii), we obtain

$$\omega_0(FX)\le(p+\overline{kh}q\psi(r_0))\omega_0(X).$$

Combining the above estimate and all properties of the operator F established before and taking into account Theorem 2.2 we complete the proof. □

Now, we are going to consider two important particular cases of Eq. (1.1) which are very often investigated in the theory of nonlinear integral equations and are applied to describe several real world problems.

The first equation is called the nonlinear functional integral equation of fractional order (cf. [18–22]). That equation has the form of Eq. (1.1) with the function $g(t,s)$ defined on the triangle Δ by the formula

$$g(t,s)=\begin{cases}\dfrac{1}{(t-s)^{1-\alpha}} & \text{for } t\neq s\\ 0 & \text{for } t=s,\end{cases}\tag{3.13}$$

where $\alpha\in(0,1)$ is a given constant.

Observe that the above defined function $g(t,s)$ satisfies assumption (vii). Indeed, it is easily seen that the function $t\to g(t,s)$ is nonincreasing on the interval $[s,1]$ for any fixed $s\in I$.

Moreover, for any fixed $t\in I$ we have

$$\int_0^t g(t,s)\mathrm{d}s=\int_0^t\frac{1}{(t-s)^{1-\alpha}}\mathrm{d}s=\frac{t^\alpha}{\alpha},$$

which means that the function $s\to g(t,s)$ is Lebesgue integrable over $[0,t]$.

In order to verify the remaining part of assumption (vii) let us fix $\varepsilon>0$ and take $t_1,t_2\in I$ such that $t_1<t_2$. Then we have:

$$\begin{aligned}\int_0^{t_1}[g(t_2,s)-g(t_1,s)]\mathrm{d}s&=\int_0^{t_1}\left[\frac{1}{(t_2-s)^{1-\alpha}}-\frac{1}{(t_1-s)^{1-\alpha}}\right]\mathrm{d}s\\&=\frac{1}{\alpha}[t_2^\alpha-t_1^\alpha-(t_2-t_1)^\alpha].\end{aligned}$$

Hence we get

$$\begin{aligned}\left|\int_0^{t_1}[g(t_2,s)-g(t_1,s)]\mathrm{d}s\right|&=\frac{(t_2-t_1)^\alpha-[t_2^\alpha-t_1^\alpha]}{\alpha}\\&\le\frac{(t_2-t_1)^\alpha}{\alpha}.\end{aligned}$$

The above estimate implies that we can take $\delta=(\alpha\varepsilon)^{1/\alpha}$ in order to ensure that condition (3.1) is satisfied.

Similarly we can show that condition (3.2) is satisfied with the same δ as above.

Further, let us pay attention to the fact that the function $h=h(t)$ defined earlier in our consideration has the form $h(t)=t^\alpha/\alpha$. This implies that $\overline{h}=1/\alpha$, where $\overline{h}$ is the constant appearing in assumption (viii).

In order to formulate our next result related to the function $g(t,s)$ defined by (3.13), let us consider the functional Volterra singular integral equation of the form

$$x(t)=f_1(t,x(t),x(a(t)))+(Gx)(t)\int_0^t\frac{k(t,s)}{(t-s)^{1-\alpha}}(Qx)(s)\mathrm{d}s,\tag{3.14}$$

for $t\in I$. Obviously Eq. (3.14) is a special case of Eq. (1.1) with the function $g(t,s)$ defined by (3.13).

Then we have the following result.

Theorem 3.6. *Assume that there are satisfied assumptions* (i)–(vi) *of Theorem* 3.5 *and assumption* (viii) *with* $\overline{h} = 1/\alpha$. *Then Eq.* (3.14) *has at least one solution in the space* $C(I)$.

Our second integral equation, which was announced earlier, is a Volterra counterpart of the famous Chandrasekhar quadratic integral equation (see [25–29], for example). That equation, after adopting to our situation, has the form

$$x(t) = f_1(t, x(t), x(a(t))) + (Gx)(t) \int_0^t \frac{tk(t, s)}{t+s} (Gx)(s)\mathrm{d}s, \tag{3.15}$$

where $t \in I = [0, 1]$.

Observe that Eq. (3.15) is a special case of Eq. (1.1) with the function $g = g(t, s)$ having the form

$$g(t, s) = \begin{cases} \dfrac{t}{t+s} & \text{for } (t, s) \in \Delta, (t, s) \neq (0, 0) \\ 0 & \text{for } t = s = 0. \end{cases} \tag{3.16}$$

In what follows we show that the function $g(t, s)$ defined by (3.16) satisfies conditions of assumption (vii). In fact, observe first that the function $g(t, s)$ is nondecreasing with respect to the variable t for any fixed s. Obviously $g(t, s) \geq 0$ for $(t, s) \in \Delta$. Moreover, for any $t \in (0, 1]$ we have

$$\int_0^t g(t, s)\mathrm{d}s = t \int_0^t \frac{\mathrm{d}s}{t+s} = t \ln 2.$$

Thus the function $s \to g(t, s)$ is Lebesgue integrable over the interval $[0, t]$ and the function $h(t)$ defined by (3.3) has the form $h(t) = t \ln 2$ for $t \in I$, $t > 0$. Obviously h is continuous on the interval I.

The above gathered facts and Lemma 3.3 allow us to infer that the function $g(t, s)$ satisfies assumption (vii). Moreover, we have that $\overline{h} = \ln 2$.

Now, we can formulate our next result.

Theorem 3.7. *Assume that there are satisfied assumptions* (i)–(vi) *of Theorem* 3.5 *and assumption* (viii) *with* $\overline{h} = \ln 2$. *Then Eq.* (3.15) *has at least one solution in the space* $C(I)$.

4. Examples

In this section we provide two examples illustrating the main result contained in Theorem 3.5 and showing its applicability.

Example 4.1. Consider the singular Volterra integral equation of the form

$$x(t) = \frac{1}{6}[te^{-t} + t^2x(t) + x(1-t)] + \frac{1}{2}\sin(t - x(t)) \int_0^t \left[t - \left(s - \frac{1}{2}\right)^2\right] g(t, s)(Qx)(s)\mathrm{d}s, \quad t \in I, \tag{4.1}$$

where

$$g(t, s) = \begin{cases} \dfrac{-1}{(t-s)\ln^3(t-s)} & \text{for } 0 \leq s < t \leq e^{-3} \\ \dfrac{1}{18}e^3 & \text{for } e^{-3} \leq s < t \leq 1 \\ 0 & \text{for } (t, s) \in \Delta, \quad t = s, \end{cases}$$

and the operator Q is defined on the space $C(I)$ in the following way

$$(Qx)(t) = \int_0^t x(\tau) \tanh(|x(\tau)|)\mathrm{d}\tau.$$

Notice that Eq. (4.1) is a particular case of Eq. (1.1), where

$$f_1(t, x, y) = \frac{1}{6}\left[te^{-t} + t^2x + y\right],$$
$$a(t) = 1 - t,$$
$$(Gx)(t) = \frac{1}{2}\sin(t - x(t)),$$
$$k(t, s) = t - \left(s - \frac{1}{2}\right)^2.$$

We can easily check that $\overline{f_1} = \frac{1}{6e}$, $\overline{k} = 1$. Moreover, we have

$$|f_1(t, x_1, y_1) - f_1(t, x_2, y_2)| \le \frac{1}{3}\max\{|x_1 - x_2|, |y_1 - y_2|\}.$$

Thus the function f_1 satisfies assumption (ii) with $p = \frac{1}{3}$. It can be also verified that there are satisfied assumptions (i), (iii)–(vi), where $q = \frac{1}{2}$, $\varphi(r) = \frac{1}{2}$, and $\psi(r) = r^2$. Apart from that, using standard tools of differential calculus it is easy to check that the function $t \to g(t, s)$ is nonincreasing on the interval $[s, 1]$, for any fixed $s \in I$. Since $g(t, s) \ge 0$ for $(t, s) \in \Delta$, in order to check that the function $s \to g(t, s)$ is Lebesgue integrable over the interval $[0, t]$ it is sufficient to show that there exists the improper integral $\int_0^t g(t, s)\mathrm{d}s$ in the classical sense. Further, we obtain the formula allowing us to express the function $h = h(t)$ defined by (3.3):

$$h(t) = \begin{cases} 0 & \text{for } t = 0 \\ \dfrac{1}{2\ln^2 t} & \text{for } 0 < t \le e^{-3} \\ \dfrac{1}{18}e^3 t & \text{for } e^{-3} < t \le 1. \end{cases}$$

Obviously the function h is continuous on I and we have that $\overline{h} = e^3/18$.

Next, fixing $t_1, t_2 \in I$ arbitrarily such that $t_1 < t_2$, we can calculate that

$$\int_{t_1}^{t_2} g(t_2, s)\mathrm{d}s = \begin{cases} \dfrac{1}{2\ln^2(t_2 - t_1)} & \text{for } 0 < t_2 \le e^{-3} \\ \dfrac{e^3}{18}(t_2 - t_1) & \text{for } e^{-3} < t_2 \le 1. \end{cases}$$

Hence we see that there is satisfied condition (3.2) of assumption (vii). Linking this fact with the continuity of the function h on the interval I and taking into account Remark 3.2 we infer that there is satisfied assumption (vii).

Finally, let us consider the first inequality appearing in assumption (viii). On the basis of the above calculations we conclude that in our situation the mentioned inequality has the form

$$\frac{1}{6e} + \frac{1}{3}r + \frac{1}{36}e^3 r^2 \le r,$$

or equivalently

$$e^3 r^2 - 24r + \frac{6}{e} \le 0.$$

We can easily seen that the number $r_0 = (12 - \sqrt{144 - 6e^2})/e^3$ is the minimal solution of this inequality. Moreover, r_0 satisfies also the inequality

$$p + \overline{k}\overline{h}q\psi(r_0) < 1,$$

occurring in the second part of assumption (viii).

Thus, in view of the above established facts and Theorem 3.5 we deduce that Eq. (4.1) has at least one solution belonging to the ball B_{r_0} in the space $C(I)$.

Example 4.2. Now we will study the following nonlinear singular Volterra integral equation

$$\begin{aligned} x(t) &= \frac{1}{3(t+1)} + \frac{t^2 \sin x(t) \cos x\left(\sqrt[3]{t^2}\right)}{4(t^4+3)} \\ &\quad + \frac{\ln(1+t^2)}{4(1+t^2)}x(t)\int_0^t \frac{t}{t+s+1}\arctan\left(\frac{1}{\sqrt{1-(t-s)^2}}\right)(Qx)(s)\mathrm{d}s, \quad t \in I, \end{aligned} \tag{4.2}$$

where the operator Q is defined on the space $C(I)$ by the formula

$$(Qx)(t) = \frac{x(t)}{1 + |x(t)|}.$$

Observe that Eq. (4.2) can be treated as a special case of Eq. (1.1) if we put

$$f_1(t, x, y) = \frac{1}{3(t+1)} + \frac{t^2 \sin x \cos y}{4(t^4+3)},$$

$$a(t) = \sqrt[3]{t^2},$$
$$(Gx)(t) = \frac{\ln(1+t^2)}{4(1+t^2)}x(t),$$
$$k(t,s) = \frac{t}{t+s+1},$$
$$g(t,s) = \begin{cases} \arctan\left(\dfrac{1}{\sqrt{1-(t-s)^2}}\right) & \text{for } (t,s) \in \Delta,\ (t,s) \neq (1,0) \\ \dfrac{\pi}{2} & \text{for } t = 1,\ s = 0. \end{cases}$$

It is easy to show that there are satisfied assumptions (i)–(iii) with $p = 1/8, \overline{f_1} = \frac{1}{3}$ and $q = \frac{1}{8}\ln 2$.

Moreover, the operators G and Q satisfy assumptions (iv) and (v), respectively, with $\varphi(r) = \frac{\ln 2}{8}r$ and $\psi(r) = r$. Obviously the function $k = k(t,s)$ satisfies assumption (vi) and $\overline{k} = \frac{1}{2}$.

Further, let us observe that using the standard methods of differential calculus we can show that the function $t \to g(t,s)$ is nondecreasing on the interval $[s, 1]$ for any fixed $s \in I$. Moreover, for $t \in I$ we get

$$\int_0^t g(t,s)\mathrm{d}s \leq \int_0^t \frac{1}{\sqrt{1-(t-s)^2}}\mathrm{d}s = \arcsin t. \tag{4.3}$$

This show that the function $s \to g(t,s)$ is Lebesgue integrable over the interval $[0, t]$. Hence we infer that the function $h = h(t)$ defined by (3.3) is well-defined. Apart from this, using the standard argumentation of the theory of integrals, we can check that the function h is continuous on the interval I.

Next, fix arbitrarily $t_1, t_2 \in I$, $t_1 < t_2$. Then we obtain

$$\int_{t_1}^{t_2} g(t_2,s)\mathrm{d}s \leq \int_{t_1}^{t_2} \frac{\mathrm{d}s}{\sqrt{1-(t_2-s)^2}} = \arcsin(t_2 - t_1).$$

Linking the above estimate with the continuity of the function h and applying Lemma 3.3 we conclude that the function $g(t,s)$ satisfies assumption (vii).

Finally, let us consider the first inequality from assumption (viii), which here has the form

$$\frac{1}{8}r + \frac{1}{3} + \frac{\ln 2}{16}\overline{h}r^2 \leq r. \tag{4.4}$$

It is rather difficult to find the exact value of the constant $\overline{h}$. However, for our purpose it is sufficient to give its evaluation which can be immediately obtained from inequality (4.3).

Indeed, we have that $\overline{h} \leq \arcsin 1 = \pi/2$.

Now, observe that a positive number r satisfies inequality (4.4) provided it satisfies the following inequality

$$\frac{1}{3} + \frac{1}{8}r + \frac{\pi \ln 2}{32}r^2 \leq r,$$

or, equivalently

$$3\pi \ln 2r^2 - 84r + 32 \leq 0.$$

It is easily seen that the number $r_0 = 0.4$ is a solution of the above inequality. Obviously, as we noticed above, this number also satisfies inequality (4.4).

On the other hand we have that the number r_0 also satisfies the second inequality $p + \overline{k}\overline{h}q\psi(r_0) < 1$ from assumption (viii).

Now, applying Theorem 3.5 we infer that Eq. (4.2) has at least one solution belonging to the ball $B_{0.4}$ in the space $C(I)$.

5. An approach through the theory of Volterra–Stieltjes integral equations

In this section we compare our results obtained in Section 3 with results which can be derived with help of the theory of Volterra–Stieltjes integral equations.

The mentioned theory and its applications to the theory of fractional integral equations was developed in the paper [6].

In order to present the main subject of the theory of Volterra–Stieltjes integral equations we introduce first a few auxiliary concepts and facts which will be used in the sequel (cf. [30]).

Assume that x is a real function defined on the interval $[a, b]$. Denote by $\bigvee_a^b x$ the variation of the function x on the interval $[a, b]$. We say that x is of bounded variation on $[a, b]$ if $\bigvee_a^b x$ is finite. In the case we have a function $u(t,s) = u$:

$[a, b] \times [c, d] \to \mathbb{R}$, then we denote by $\bigvee_{t=p}^{q} u(t, s)$ the variation of the function $t \to u(t, s)$ on the interval $[p, q] \subset [a, b]$. Similarly we define the quantity $\bigvee_{s=p}^{q} u(t, s)$.

If x and φ are real functions defined on the interval $[a, b]$ then under some additional conditions [30] we can define the Stieltjes integral (in the Riemann–Stieltjes sense)

$$\int_a^b x(t)\mathrm{d}\varphi(t)$$

of the function x with respect to the function φ. In such a case we will say that x is Stieltjes integrable on the interval $[a, b]$ with respect to φ. Notice that several conditions are known guaranteeing the existence of the Stieltjes integral. One of the most frequently used requires [30] that x is continuous and φ of bounded variation on $[a, b]$.

We can also consider the Stieltjes integral of the form

$$\int_a^b x(s)d_s g(t, s),$$

where $g : [a, b] \times [c, d] \to \mathbb{R}$ and the symbol d_s indicates the integration with respect to s. In such a case we have to impose conditions ensuring both the existence of such an integral and its applicability in the theory of the so-called Urysohn–Stieltjes or Volterra–Stieltjes integral equations (cf. [6,31]).

In below formulated result we present assumptions of such a type. For simplicity we will assume that $[a, b] = I = [0, 1]$.

The draft of the result we provide now comes from the paper [6] but we formulate it in such a form to connect it with Eq. (1.1) considered in this paper.

Thus, let us consider the nonlinear Volterra–Stieltjes integral equation having the form

$$x(t) = f_1(t, x(t), x(a(t))) + (Gx)(t) \int_0^t k(t, s)(Qx)(s)d_s h(t, s), \tag{5.1}$$

for $t \in I$. We will assume that the functions f_1, a, k and the operators G and Q satisfy the same assumptions as those imposed in Theorem 3.5, i.e. assumptions (i)–(v) and that part of assumption (vi) which requires that $k : \Delta \to \mathbb{R}$ be continuous on Δ. Instead of assumption (vii) we require that the following hypotheses concerning the function $h = h(t, s)$ be satisfied (cf. [6, Theorem 2]):

(ix) The function $h : \Delta \to \mathbb{R}_+$ is continuous on the triangle Δ.

(x) The function $s \to h(t, s)$ is of bounded variation on the interval $[0, t]$ for any fixed $t \in I$.

(xi) For any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $t_1, t_2 \in I$ with $t_1 < t_2$ and $t_2 - t_1 \le \delta$ the following inequality holds

$$\bigvee_{s=0}^{t_1} [h(t_2, s) - h(t_1, s)] \le \varepsilon.$$

(xii) $h(t, 0) = 0$ for any $t \in I$.

In what follows we will not formulate a counterpart of assumption (viii) of Theorem 3.5 but we restrict ourselves to comparing assumptions (ix)–(xii) listed above with assumption (vii) of Theorem 3.5. Namely we show that under some additional assumption Eq. (1.1) can be treated as a particular case of Eq. (5.1).

To this end assume that the function $g(t, s) = g : \Delta \to \mathbb{R}_+$ satisfies assumption (vii) of Theorem 3.5. Apart from this we will additionally assume that the function $s \to g(t, s)$ is continuous on the interval $[0, t]$ for any fixed $t \in I$.

Further, consider the function $h : \Delta \to \mathbb{R}_+$ defined by the formula

$$h(t, s) = \int_0^s g(t, z)\mathrm{d}z. \tag{5.2}$$

Obviously the function $h = h(t, s)$ is well-defined on the triangle Δ. Moreover, the continuity of the function $s \to g(t, s)$ on $[0, t]$ implies that $d_s h(t, s) = g(t, s)\mathrm{d}s$. This shows that in the considered situation Eq. (5.1) coincides with Eq. (1.1).

Further we show that the function $h(t, s)$ defined by (5.2) satisfies assumptions (x)–(xii) formulated above. Moreover, it also "partly" satisfies assumption (ix).

First, observe that in view of nonnegativity of $g(t, z)$ on Δ we infer that the function $s \to h(t, s)$ is nondecreasing on the interval $[0, t]$ for each fixed $t \in I$. Thus the function h satisfies assumption (x).

Next, fix $\varepsilon > 0$ and choose $\delta > 0$ according to conditions (3.1) and (3.2) of assumption (vii). Then we have:

$$\begin{aligned} \bigvee_{s=0}^{t_1} [h(t_2, s) - h(t_1, s)] &= \bigvee_{s=0}^{t_1} \left[\int_0^s g(t_2, z)\mathrm{d}z - \int_0^s g(t_1, z)\mathrm{d}z \right] \\ &= \bigvee_{s=0}^{t_1} \int_0^s [g(t_2, z) - g(t_1, z)]\mathrm{d}z. \end{aligned} \tag{5.3}$$

Further, keeping in mind that the function $t \to g(t, s)$ is monotonic on the interval $[s, 1]$, we consider two cases.

First, assume that this function is nondecreasing. Then $g(t_2, z) - g(t_1, z) \geq 0$ which implies that the function

$$s \to \int_0^s [g(t_2, z) - g(t_1, z)]\mathrm{d}z$$

is nondecreasing on $[0, t_1]$. Linking this fact with (5.3) we obtain

$$\bigvee_{s=0}^{t_1} [h(t_2, s) - h(t_1, s)] = \int_0^{t_1} [g(t_2, z) - g(t_1, z)]\mathrm{d}z \leq \varepsilon,$$

which is a consequence of condition (3.1). Thus assumption (xi) is satisfied in this case.

If we assume that the function $t \to g(t, s)$ is nonincreasing on $[s, 1]$ then we have that $g(t_2, s) - g(t_1, s) \leq 0$. This implies that the function

$$s \to \int_0^s [g(t_2, z) - g(t_1, z)]\mathrm{d}z,$$

is nonincreasing on $[0, t_1]$. Hence, in view of (5.3) we obtain by (3.1):

$$\begin{aligned} \bigvee_{s=0}^{t_1} [h(t_2, s) - h(t_1, s)] &= \int_s^0 [g(t_2, z) - g(t_1, z)]\mathrm{d}z \\ &= \left| \int_0^s [g(t_2, z) - g(t_1, z)]\mathrm{d}z \right| \leq \varepsilon. \end{aligned}$$

Thus, also in the second case we see that assumption (xi) is satisfied.

Obviously we have that $h(t, 0) = 0$, so (xii) is also trivially satisfied.

Finally, we pay our attention to assumption (ix). To this end observe that from the fact that the function $g(t, s)$ satisfies condition (3.2) of assumption (vii) we deduce that for $\varepsilon > 0$ there exists $\delta > 0$ such that if $t_1, t_2 \in I$, $t_1 < t_2$ and $t_2 - t_1 \leq \delta$, then

$$\int_{t_1}^{t_2} g(t_2, z)\mathrm{d}z \leq \varepsilon.$$

Hence we have

$$\begin{aligned} \int_{t_1}^{t_2} g(t_2, z)\mathrm{d}z &= \left| \int_{t_1}^{t_2} g(t_2, z)\mathrm{d}z \right| = \left| \int_0^{t_2} g(t_2, z)\mathrm{d}z - \int_0^{t_1} g(t_2, z)\mathrm{d}z \right| \\ &= |h(t_2, t_2) - h(t_2, t_1)| \leq \varepsilon. \end{aligned}$$

This observation shows that in the paper [6] instead of assumption (ix) it is sufficient to require the continuity of the function $h = h(t, s)$ with respect to the variable s only. Hence we also conclude that our results are more general than those obtained in [6].

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